Projective production of path-entangled photon number states with linear optics

Hwang Lee, Pieter Kok, Nicolas J. Cerf and Jonathan P. Dowling
Quantum Computing Technologies Group, Exploration Systems Autonomy, Section 367
Jet Propulsion Laboratory, California Institute of Technology
MS 126-347, 4800 Oak Grove Drive, Pasadena, CA 91109-8099
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We propose a method for preparing path entangled states with a definite photon number larger than two that relies on projective measurements. In contrast with the previously known schemes, our method uses linear optics elements only. Specifically, we exhibit a way of generating four-photon path-entangled states of the form $|4,0\rangle + |0,4\rangle$ requiring four beam splitters and two detectors. These states are of major interest as a resource for quantum interferometric optical lithography as well as for quantum interferometric sensors.

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Quantum entanglement plays a central role in quantum communication and computation. It also provides a significant improvement in frequency standards as well as in the performance of interferometric sensors [1,2]. In the latter context, it has been shown that the Heisenberg limit of phase sensitivity of a Mach-Zehnder interferometer can be reached by using maximally entangled states with a definite number of photons N, that is, states of the form

$$|N,0\rangle_{A,B} + |0,N\rangle_{A,B} \tag{1}$$

with A and B denoting the two arms of the interferometer. These states, also called path-entangled photon number states, give rise to a phase sensitivity of order $1/N$, whereas using coherent light in the interferometer gives a corresponding sensitivity of order $1/\sqrt{N}$, with N being the average number of photons. Another potential application of quantum entanglement is photolithography. Indeed, it has been shown recently that the Rayleigh diffraction limit in optical lithography can be beaten by the use of path-entangled photon number states [3]. In order to obtain the N-fold resolution enhancement with quantum interferometric optical lithography, one needs again to create an N-photon path-entangled state of the form (1), where A and B are two distinct paths. Due to interference of the paths one obtains an intensity pattern at the lithographic surface which is proportional to $\cos N\varphi$, where φ parametrizes the position on the surface. An (incoherent) superposition of these states with varying N and suitable phase shifts then yields a Fourier series up to a constant [4]. Finding methods for generating these path-entangled states has been a longstanding endeavour in quantum optics. Unfortunately, with the notable exception of $N = 2$, the optical generation of these states has been known to require a large nonlinear interaction, typically a Kerr element, which makes any physical implementation very hard [5–7].

Recently, however, several methods have been proposed for the realization of probabilistic quantum logic gates that make use of linear optics and projective measurements [8–10], namely by measuring some part of the system while the rest of it is projected onto a desired state (also called state reduction). Since the state is obtained conditioned on a measurement outcome, this method only works probabilistically. In this letter, we investigate a new technique for generating path-entangled photon number states based on this new paradigm. We suggest several linear optics schemes based on projective measurements for the preparation of four-photon path-entangled states. Finally we discuss the feasibility of these schemes by investigating the consequence of inefficient detectors on the state preparation process.

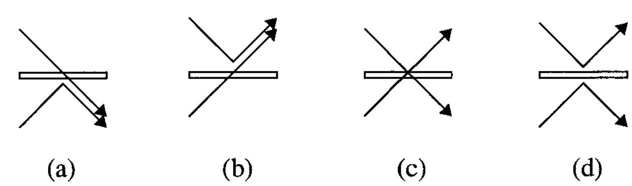

FIG. 1. Four possibilities obtained by sending a $|1,1\rangle$ state through a beam splitter. The diagrams (c) and (d) lead to the same final state, and interfere destructively. (c) transmission-transmission $(i)(i) = -1$; (d) refection-reflection: (-1)(-1)=1.

It is well known that two-photon path-entangled states ($N = 2$) can be created using a Hong-Ou-Mandel interferometer, where a photon pair from a parametric down-converter impinge onto a 50:50 beam splitter [11]. The beam splitter operation on the product state $|1,1\rangle_{A,B}$ yields the path-entangled state $|2,0\rangle_{A',B'} + |0,2\rangle_{A',B'}$. The probability amplitude for having $|1,1\rangle_{A',B'}$ at the output of the beam splitter vanishes, which can be understood by a simple diagrammatic analysis (see Fig. 1). In our convention, the reflected mode acquires a phase -1 while the transmitted mode acquires a phase of i (this must be consistent with the reciprocity requirement), so that the two possible ways of having a state $|1,1\rangle$ interfere destructively. Interestingly enough, it can be proven that a simple beam splitter is not sufficient for the production of path-entangled states with a photon number larger than two [12]. For $N > 2$, common knowledge is that nonlinear optical elements become necessary. By contrast, we will show here that the recourse to nonlinear elements can actually be avoided if some photo-detectors are added to the scheme. The desired path-entangled state is then obtained conditioned on the measurement outcome.

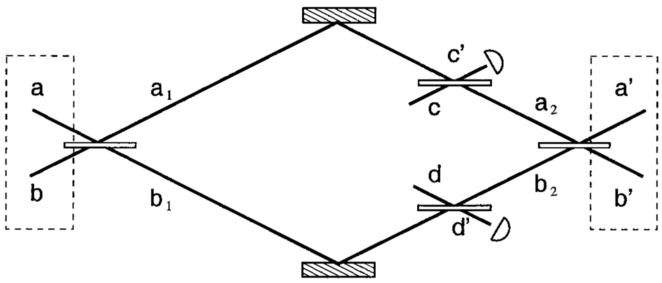

FIG. 2. Mach-Zehnder interferometer configuration with two additional beam splitters in the lower and upper arms, which direct the reflected beams to photodetectors. The measurement allows the projective generation of the states $|2,0\rangle + |0,2\rangle$ and $|4,0\rangle + |0,4\rangle$.

Before considering the case of $N = 4$, it is instructive to exhibit first the generation of the state $|2,0\rangle_{A',B'} + |0,2\rangle_{A',B'}$ using projective measurements instead of a simple beam splitter. Let us consider a Mach-Zehnder interferometer with two additional beam splitters, each of them being followed by a detector (see Fig. 2). In such a configuration, one can post-select the desired state on a two-fold detector coincidence. Formally, we are dealing with a four-port optical device wich may be represented by expressing the output mode operators $\hat{a}'$, $\hat{b}'$, $\hat{c}'$, and $\hat{d}'$ as a function of the input mode operators $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$ [2]. For the transformation of a single beam splitter (say the first one, see Fig. 2), we use the convention

$$\begin{aligned} \hat{a}_1 &= (-\hat{a} + i\hat{b})/\sqrt{2} \\ \hat{b}_1 &= (i\hat{a} - \hat{b})/\sqrt{2} \end{aligned} \tag{2}$$

Combining the transformations for the first, last, and the two intermediate beam splitters in the lower and upper arms, we get the overall transformation

$$\begin{aligned} \hat{a}' &= \hat{b}/\sqrt{2} + (\hat{c} - i\hat{d})/2 \\ \hat{b}' &= \hat{a}/\sqrt{2} + (\hat{d} - i\hat{c})/2 \\ \hat{c}' &= (\hat{a} - i\hat{b})/2 + i\hat{c}/\sqrt{2} \\ \hat{d}' &= (\hat{b} - i\hat{a})/2 + i\hat{d}/\sqrt{2} \end{aligned} \tag{3}$$

(Note that we neglect the phase induced by the mirrors and the optical paths.) For a given input state, one can obtain the output state simply by expressing the input mode operators in terms of the output modes, that is, by inverting Eq. (3). Suppose the input state is $|2,2\rangle_{A,B} = \frac{1}{4}(\hat{a}^\dagger)^2(\hat{b}^\dagger)^2|0\rangle$. Then, the term of order $\hat{c}'\hat{d}'$ in the expansion of $(\hat{a}^\dagger)^2(\hat{b}^\dagger)^2$ can be shown to be $\frac{-i}{4}(\hat{a}'^{\dagger 2} + \hat{b}'^{\dagger 2})$ so that the total output state after post-selection can be written as

$$|2,0\rangle_{A',B'} + |0,2\rangle_{A',B'} , \tag{4}$$

where we have only shown the term that corresponds to the measurement of one photon at each detector. Thus, if one and only one photon is detected at each detector, one obtains the envisioned two-photon path-entangled output state. The probability of this event is $1/16$.

The reason why this projective method works can be understood as follows. The state after passing through the first beam splitter becomes a linear superposition of the states $|4,0\rangle$, $|2,2\rangle$, and $|0,4\rangle$. Again, the states $|3,1\rangle$ and $|1,3\rangle$ do not appear for the same reason as the vanishing of the state $|1\rangle'_A|1\rangle'_B$ when the input is $|1\rangle_A|1\rangle_B$ (see, Fig 3). Since the detection of one photon at each detector requires at least one photon in both the upper and lower arms of the interferometer, the $|4,0\rangle$ and $|0,4\rangle$ states cannot contribute to such events. Consequently, only the $|2,2\rangle$ term is left, which yields $|1,1\rangle$ since one photon is detected in each arm. This last state is thus found at the input ports of the last beam splitter, which results in expected state $|2,0\rangle + |0,2\rangle$.

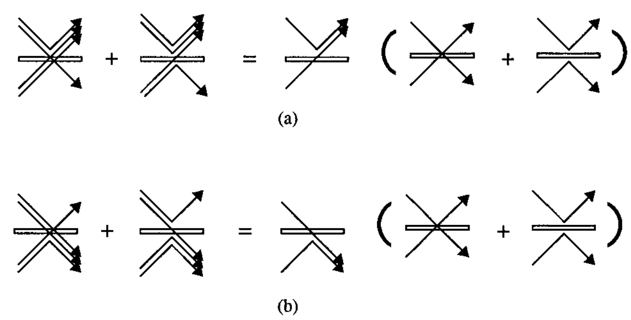

FIG. 3. Two possible ways of making a $|3,1\rangle$ (and $|1,3\rangle$) state from an input $|2,2\rangle$ state passing through a beam splitter. The two diagrams interfere destructively just as in Fig. 1.

We can now use this to proceed to the generation of the $|4,0\rangle + |0,4\rangle$ state. The key reason why projective measurement is useful in the above scheme is that it enables us to conditionally suppress the extreme components $|4,0\rangle$ and $|0,4\rangle$ in the interferometer, while leaving the middle component $|2,2\rangle$ unchanged. More generally, we will see that the generation of path-entangled states with $N > 2$ requires eliminating (or reducing the amplitude of) the extreme components with respect to the central ones. Suppose we want to produce the state $|4,0\rangle + |0,4\rangle$. Then, what must be the input state of the last beam splitter? A simple matrix inversion shows that one needs an input operator as $[(\hat{a}^\dagger)^4 - 6(\hat{a}^\dagger)^2(\hat{b}^\dagger)^2 + (\hat{b}^\dagger)^4]$. On the other hand, for the output state $|4,0\rangle - |0,4\rangle$, the required input operator is $[(\hat{a}^\dagger)^3(\hat{b}^\dagger) - (\hat{a}^\dagger)(\hat{b}^\dagger)^3]$. (Since it has fewer terms, we will focus for the moment on producing $|4,0\rangle - |4,0\rangle$.) The required input state of the last beam splitter should now be compared to the output state of the first beam splitter. Taking $|3,3\rangle$ as the input state, the first beam splitter transforms this into a linear superposition of $|6,0\rangle$, $|4,2\rangle$, $|2,4\rangle$, and $|0,6\rangle$ such that

$$(\hat{a}^\dagger\hat{b}^\dagger)^3 \Rightarrow (\hat{a}^\dagger)^6 + 3(\hat{a}^\dagger)^4(\hat{b}^\dagger)^2 + 3(\hat{a}^\dagger)^2(\hat{b}^\dagger)^4 + (\hat{b}^\dagger)^6 \tag{5}$$

After passing through the intermediate beam splitters and if one photon is detected at each detector (*one click at both detectors*), the corresponding quantum state is then projected on to $|3,1\rangle + |1,3\rangle$. Indeed, since the states $|6,0\rangle$ or $|0,6\rangle$ cannot yields a click at *both* detectors, they are eliminated in this projective process. The $|4,2\rangle$ and $|2,4\rangle$ states, on the other hand, lose one photon in each arm of the interferometer, and are therefore reduced to $|3,1\rangle$ and $|1,3\rangle$, respectively. Thus, just before the last beam splitters, we have $|3,1\rangle + |1,3\rangle$. In order to match the relative phase of the state (for $|4,0\rangle - |0,4\rangle$, we need $|3,1\rangle - |1,3\rangle$), it is sufficient to use a phase shifter (see Fig. 3). With a $\pi/2$ phase shift in the lower arm of the interferometer, the state before the intermediate beam splitters, is given as $|6,0\rangle + 3e^{i2\pi/2}|4,2\rangle + 3e^{i4\pi/2}|2,4\rangle + e^{i6\pi/2}|0,6\rangle$, which is then reduced to $|3,1\rangle - |1,3\rangle$. Consequently after the last beam splitter, we get $|4,0\rangle - |0,4\rangle$. The probability to obtain this a state as $3/64 \approx 0.05$. Any $|2N+1, 2N+1\rangle$ input state with detection of $2N-1$ photons at each detector yields the output state $|4,0\rangle - |0,4\rangle$ in this configuration, but with smaller probabilities as N increases.

There is another way to produce $|4,0\rangle + |0,4\rangle$, exploiting the previously unused input ports. Consider the input state $|2,2\rangle$ at modes A and B, and let $|2,0\rangle + |0,2\rangle$ be the incoming state at the other two input ports (c, d modes in Fig. 2). Again conditioned on a two-fold single-photon detection coincidence this protocol yields the state $|4,0\rangle - |0,4\rangle$ with probability $3/64$.

So far, our schemes for generation of $|N,0\rangle + |0,N\rangle$ state ($N = 2,4$) entirely rely on an *a priori* symmetric $|M,M\rangle$ input state. We now develop a scheme in which we do not necessarily need such a symmetric product state. Suppose an input state $|3,0\rangle_{A,B}$. The first beam spliter (in Fig. 3) yields

$$\hat{a}^{\dagger 3} \Rightarrow \frac{-i}{\sqrt{8}}\left[\hat{a}^{\dagger 3} - 3i\hat{a}^{\dagger 2}\hat{b}^\dagger - 3\hat{a}^\dagger\hat{b}^{\dagger 2} + i\hat{b}^{\dagger 3}\right]. \tag{6}$$

Now at the intermediate BS let us feed with $|1,0\rangle_{CD} + |0,1\rangle_{CD}$ state (which is an one-photon state after $|1\rangle$ state passing through a BS). Then, we can write

$$|3,0\rangle_{A_1B_1}(|1,0\rangle_{CD} + |0,1\rangle_{CD}) \equiv |3,1\rangle_{A_1C}|0,0\rangle_{B_1D} + |3,0\rangle_{A_1C}|0,1\rangle_{B_1D} \tag{7}$$

The first term in the right hand side does not give a click at the lower detector, and the second term contains (after the imtermediate BS) $|2,1\rangle_{A_2C'}|0,1\rangle_{B_2D'}$ state with a phase factor of $(-1)(-3/4)$. Detection of one photon at each detector yields $|2,0\rangle_{A_2B_2}$. Similarly, for $|0,3\rangle_{A_1B_1}$ we have $|0,2\rangle_{A_2B_2}$ with phase factor $(-i)(3i/4)$.

On the other hand, for $|2,1\rangle_{A_1B_1}$ state, it can be rearranged as

$$|2,1\rangle_{A_1C}|1,0\rangle_{B_1D} + |2,0\rangle_{A_1C}|1,1\rangle_{B_1D}. \tag{8}$$

As depicted in Fig. 1, $|1,1\rangle_{B_1D}$ state in the second term can not yield *only one click* at the lower detector, and the first term yields the $|2,0\rangle_{A_2B_2}|1,1\rangle_{C'D'}$ state with phase factor of $(3i)(-1/4)$. Similarly, from $|1,2\rangle_{A_1B_1}$ state we have $|0,2\rangle_{A_2B_2}|1,1\rangle_{C'D'}$ state with phase factor of $3(-i/4)$.

Including the overall phase factor $1/\sqrt{8}$ in Eq. (10), one photon detection at each detector yields then, project the final state $|2,0\rangle + |0,2\rangle$, with the probability $3/64$.

Note that the desired $|2,0\rangle + |0,2\rangle$ is produced before the last BS. In order to complete the interferometer scheme (it is not at all necessary, but is useful for later discussions), we insert a $\pi/2$-phase shifter in the lower arm just after the first BS. When we do that, we obtain $|2,0\rangle - |0,2\rangle$ at the interferometer output.

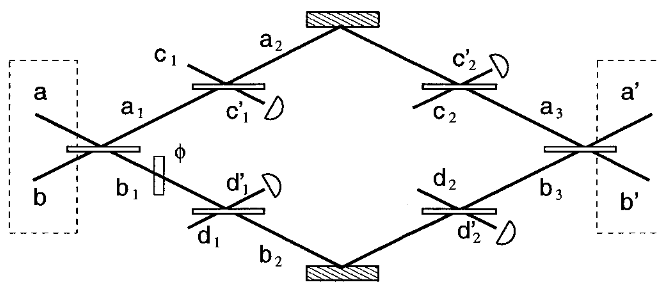

FIG. 4. Four-detector scheme with a Mach-Zehnder interferometer in order to generate the $|4,0\rangle + |0,4\rangle$ state.

Generation of the $|2,0\rangle - |0,2\rangle$ state is also possible with the four-detector cofiguration with the input state $|5,0\rangle$ by feeding the first intermediate BS with $|1,0\rangle_{C_1D_1} + |0,1\rangle_{C_1D_1}$. We find that the probability of success is even slightly higher than two-detector scheme as $(5/4)(3/64) \approx 0.06$. An interesting extension can be found in the following way: For the input state $|5,0\rangle$, if we feed the first intermediate BS with $|2,0\rangle_{C_1D_1} + |0,2\rangle_{C_1D_1}$, The one photon detection at every four detectors yields the state $|3,0\rangle - |0,3\rangle$. The probability of success in this case is approximately 5×10^{-3}. Repeatedly, when we prepare the first intermediate BS input with $|3,0\rangle_{C_1D_1} + |0,3\rangle_{C_1D_1}$, we can generate $|4,0\rangle - |0,4\rangle$. The probability of success in this case is approximately 10^{-3}.

What happens when we use realistic detectors? There are three classes of errors which will affect the outgoing state. First, the detector might register a photon only part of the time. The efficiency of the detector is then given by the probability η^2 of a succesful detection. Secondly, it might give a 'dark' count when no photon was actually present. This type of errors is usually taken to be negligable. Finally, the detector might not be able to resolve one or more photons. Since the above protocol relies on 1-, 2- and 3-photon counts, we must use single-photon resolution detectors.

We can model the detector efficiency as an ideal detector preceded by a beam splitter with transmission amplitude η. The photons which are deflected from the detector represent the loss. When two photons enter the inefficient detector, one of them might be lost, thus yielding an incorrect detector outcome. This is particulary important to our scheme, since we condition the outgoing state on single-photon detection events. The POVM for a single-photon detection is therefore given by

$$\hat{E}_1 = \sum_{n=0}^{\infty} n\eta^2(1-\eta^2)^{n-1}|n\rangle\langle n| \, . \tag{9}$$

When we apply this to the creation of the state $|4,0\rangle_{a'b'} - |0,4\rangle_{a'b'}$, we obtain $\rho^{\rm out}_{a'b'} \propto \sum_{n,m=2}^{6} nm\eta^4(1-\eta^2)^{n+m-2}\rho^{(n,m)}_{a'b'}$, where n, m are the number of photons lost in modes c' and d', and $\rho^{(n,m)} \propto_{c'd'} \langle n,m|\rho_{a'b'c'd'}|n,m\rangle_{c'd'}$. The density matrices $\rho_{(n,m)}$ which arise due to these imperfect detections again correspond to N-photon path entanglement:

(n,m)	$\rho^{(n,m)}$
$(1,2)$	$\|3,0\rangle + \|0,3\rangle$
$(2,2)$	$\|2,0\rangle + \|0,2\rangle$
$(3,1)$	$\|2,0\rangle + \|0,2\rangle$
$(3,2)$	$\|1,0\rangle + \|0,1\rangle$
$(4,1)$	$\|1,0\rangle + \|0,1\rangle$

For a realistic single-photon resolution photo-detector with efficiency $\eta^2 = 0.88$ [13], the fidelity of the outgoins state with respect to the intended state $|\Psi\rangle = |4,0\rangle + |0,4\rangle$ conditioned on a single-photon detector coincidence would be

$$F = \langle\Psi|\rho|\Psi\rangle = 0.64 \, . \tag{10}$$

Even though these imperfect detections lead to a degraded fidelity with respect to the envisioned state $|4,0\rangle - |0,4\rangle$, we can exploit the POVM's to create (incoherent) superpositions of path-entangled photons. Recall that the state $|N,0\rangle + |0,N\rangle$ gives rise to a deposition rate $1+\cos(N\varphi)$ on the substrate. Superposing these patterns with suitable intensities for different N yields a Fourier series up to a constant. This is useful for the pseudo-Fourier method in quantum lithography [4]. Note that we do not need a coherent superposition of these states since there is no interference between different photon number states.

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[1] J.J. Bollinger, W.M. Itano, and D.J. Wineland, Phys. Rev. A **54**, R4649 (1996).
[2] J.P. Dowling, Phys. Rev. A **57**, 4736 (1998).
[3] A.N. Boto, et al., Phys. Rev. Lett. **85**, 2733 (2000).
[4] P. Kok *et al.*, Phys. Rev. A **63**, 063407 (2001);
[5] G.J. Milburn, Phys. Rev. Lett. **62**, 2124 (1989).
[6] J.D. Franson and T.B. Pittman, Phys. Rev. A **60**, 917 (1999).
[7] C. Gerry and R.A. Campos, Phys. Rev. Lett. (in press) (2001).
[8] E. Knill, R. Laflamme, and G.J. Milburn, Nature **409**, 46 (2001).
[9] M. Koashi, T. Yamamoto, and N. Imoto, Phys. Rev. A **63**, 030301-1 (2001).
[10] T.B. Pittman, B.C. Jacobs, and J.D. Franson, quant-ph/0107091
[11] C.K. Hong, Z.Y. Ou, and L. Mandel, Phys. Rev. Lett. **59**, 2044 (1987).
[12] R.A. Campos, B.E.A. Saleh, and M.C. Teich, Phys. Rev. A **40**, 1371 (1989).
[13] S. Takeuchi, J. Kim, Y. Yamamoto and H. H. Hogue, Applied Phys. Lett. **74**, 1063 (1999); J. Kim, S. Takeuchi, Y. Yamamoto and H. H. Hogue, Applied Phys. Lett. **74**, 902 (1999);